The simplex algorithm and the Hirsch conjecture: **Lecture 3**

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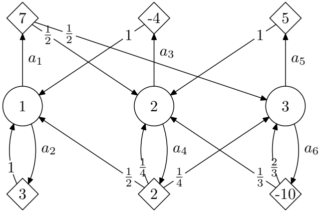
MADALGO & CTIC Summer School

August 11, 2011

Overview

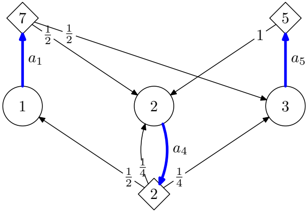
- **Lecture 1:**
  - Introduction to linear programming and the simplex algorithm.
  - Pivoting rules.
  - The RandomFacet pivoting rule.
- **Lecture 2:**
  - The Hirsch conjecture.
  - Introduction to Markov decision processes (MDPs).
  - Upper bound for the LargestCoefficient pivoting rule for MDPs.
- **Lecture 3:**
  - Lower bounds for pivoting rules utilizing MDPs. Example: Bland's rule.
  - Lower bound for the RandomEdge pivoting rule.
  - Abstractions and related problems.

Discounted Markov decision processes



$$J = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \qquad P = \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ 1 & 0 & 0 \\ 1 & 0 & 0 \\ \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ 0 & 1 & 0 \\ 0 & \frac{1}{3} & \frac{2}{3} \end{bmatrix} \qquad c = \begin{bmatrix} 7 \\ 3 \\ -4 \\ 2 \\ 5 \\ -10 \end{bmatrix}$$

- A discounted MDP with n states and a total of m actions can be represented by:
    - A discount factor $\gamma < 1$.
    - A zero-one matrix $J \in \{0,1\}^{m \times n}$, with $J_{a,i} = 1$ iff $a \in A_i$.
    - A stochastic matrix $P \in \mathbb{R}^{m \times n}$.
    - A reward vector $c \in \mathbb{R}^m$.



$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad P_\pi = \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ 0 & 1 & 0 \end{bmatrix} \qquad c_\pi = \begin{bmatrix} 7 \\ 2 \\ 5 \end{bmatrix}$$

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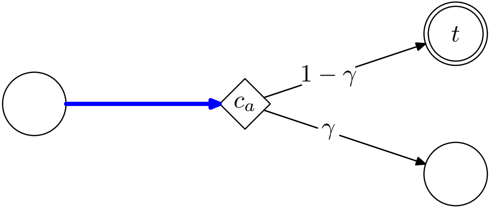
  - A stochastic matrix $P \in \mathbb{R}^{m \times n}$.
  - A reward vector $c \in \mathbb{R}^m$.
- A policy π is a choice of an action from each state. π defines a Markov chain with rewards (P_π, c_π).

The stopping condition



- The discount factor $\gamma < 1$ was introduced because the expected total reward $\sum_{k=0}^{\infty} b^T P_\pi^k c_\pi$, where b is some initial distribution, may not converge.
- For every action a, $(1-\gamma)$ may be interpreted as the probability of moving to a **terminal state** t.

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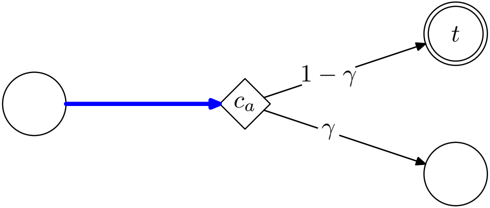

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- For every action a, $(1-\gamma)$ may be interpreted as the probability of moving to a **terminal state** t.
- To ensure convergence it is enough to satisfy the following condition:
  - **Stopping condition:** The terminal state is eventually reached with probability 1 from all states.

The stopping condition

- Let $P_\pi \in \mathbb{R}^{n \times n}$ be a matrix with non-negative entries such that each row sums to at most 1.
  - The difference between 1 and the sum of the a'th row is the probability of moving to the terminal state when using action a.
  - Note that γP, where P is an $n \times n$ stochastic matrix, is a special case.

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- If the stopping condition is satisfied, each row of P_π^n sums to less than 1, and $P_\pi^k \to 0$ for $k \to \infty$.
- It is again not difficult to show that:

$$(I - P_\pi)^{-1} = \sum_{k=0}^{\infty} P_\pi^k$$

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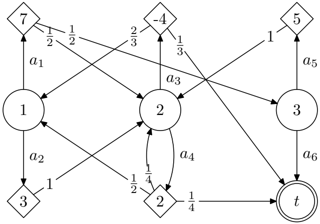
- Everything that was said in lecture 2 about discounted Markov chains with rewards remain true if γP is replaced by P, where P satisfies the stopping condition.

Markov decision processes



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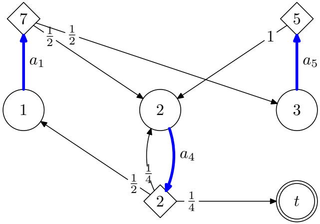

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  - A reward vector $c \in \mathbb{R}^m$.
- An MDP satisfies the stopping condition if all policies π satisfy the stopping condition. For simplicity, we will generally assume that MDPs satisfy the stopping condition.

- Every policy π defines value and flux vectors:

$$v_\pi = (I - P_\pi)^{-1} c_\pi \qquad x_\pi^T = e^T (I - P_\pi)^{-1}$$

where $(I - P_\pi)^{-1} = \sum_{k=0}^{\infty} P_\pi^k$.

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- The value of state i, $(v_\pi)_i$, is the expected total reward accumulated when starting there.
- A policy π^* is **optimal** if it maximizes the values of all states: $v_{\pi^*} \geq v_\pi$ for all π.

Markov decision processes

- An optimal policy can be found by solving a linear program:

$$(P)\quad \begin{array}{lrcl} \max & c^T x & & \\ s.t. & (J-P)^T x & = & e \\ & x & \geq & 0 \end{array} \qquad (D)\quad \begin{array}{lrcl} \min & e^T y & & \\ s.t. & (J-P)y & \geq & c \end{array}$$

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- There is a one-to-one correspondence between policies and basic feasible solutions of the primal LP (P).
- The reduced cost vector, i.e. the coefficients of a tableau, $\bar{c}^{\pi} \in \mathbb{R}^m$ for a policy π is defined by:

$$\forall i \in S, \forall a \in A_i : \quad \bar{c}_a^{\pi} = (c_a + P_a v_{\pi}) - (v_{\pi})_i$$

- $\bar{c}_a^{\pi}$ is the improvement over the current value by using a for one step w.r.t. v_{π}.

- If $\bar{c}_a^\pi > 0$ we say that a is an **improving switch** w.r.t. π. I.e., $a \in A_i$ is an improving switch iff:

$$(v_\pi)_i < c_a + P_a v_\pi$$

Lemma (Howard (1960))

Let π' be obtained from π by jointly performing any non-empty set of improving switches. Then $v_{\pi'} \geq v_\pi$ and $v_{\pi'} \neq v_\pi$.

Lemma (Howard (1960))

A policy π is optimal iff there are no improving switches.

Function $\textsc{PolicyIteration}(\pi)$

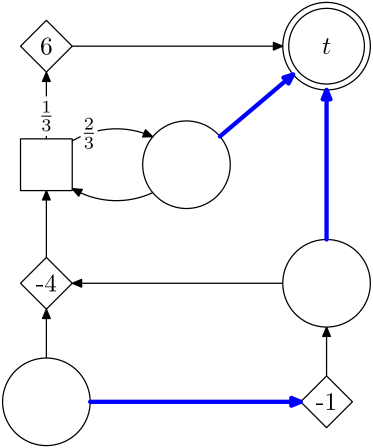
while $\exists$ *improving switch w.r.t.* π **do**
    Update π by performing improving switches

return π

- The simplex algorithm applied to the primal LP (P) is a special case of $\textsc{PolicyIteration}$.

Example: A simple MDP

- Notation for graphical representation:
  - Circles are states.
  - Diamond-shaped vertices are rewards.
  - Squares are randomization vertices.
- A **policy** π is shown as bold blue arrows.



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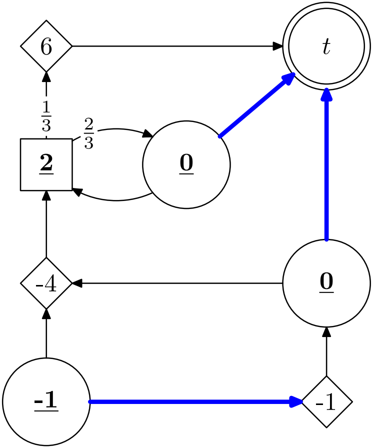

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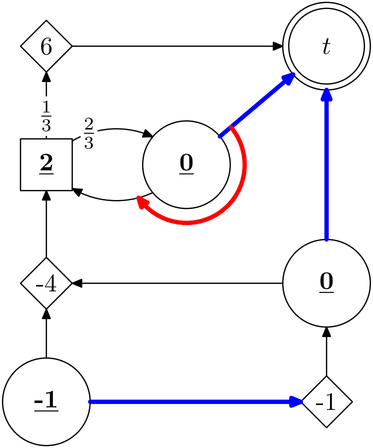

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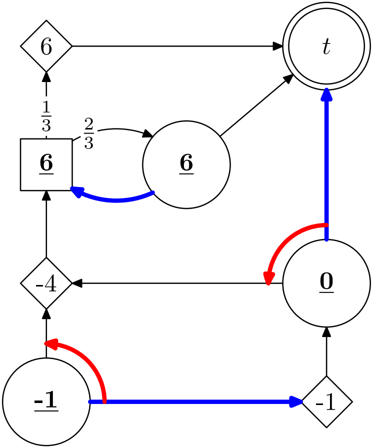

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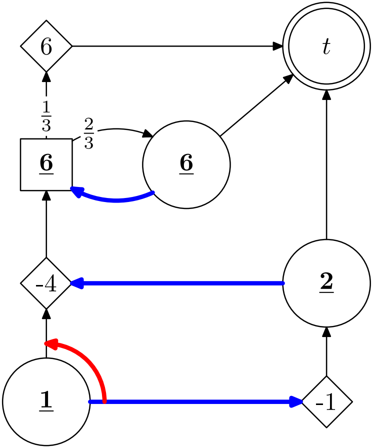

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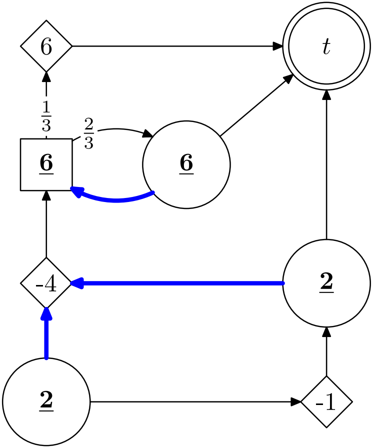
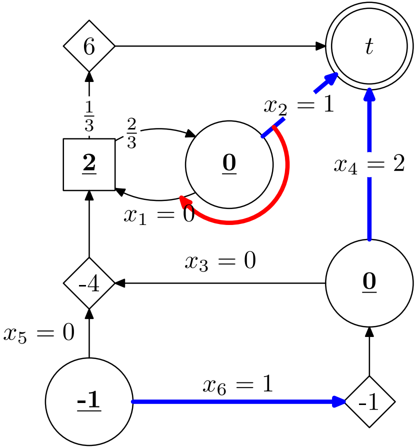
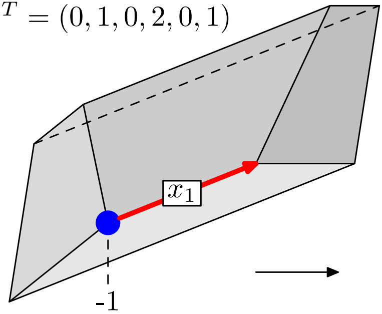


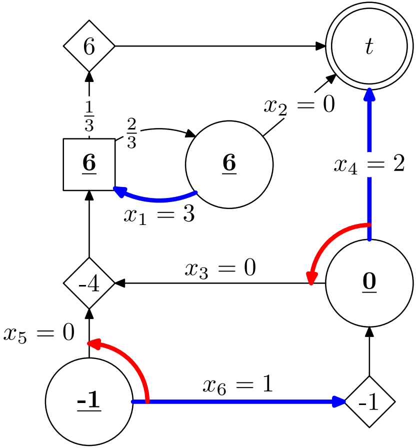


$$\begin{aligned} \max \quad & -1 + 2x_1 - 2x_3 - x_5 \\ \text{s.t.} \quad & x_2 = 1 - \tfrac{1}{3}x_1 + \tfrac{2}{3}x_3 + \tfrac{2}{3}x_5 \\ & x_4 = 2 - x_3 - x_5 \\ & x_6 = 1 - x_5 \\ & x_1, x_2, x_3, x_4, x_5, x_6 \geq 0 \end{aligned}$$

$$x^T = (0, 1, 0, 2, 0, 1)$$



From MDP to LP



$$
\begin{aligned}
\max \quad & 5 - 6x_2 + 2x_3 + 3x_5 \\
\text{s.t.} \quad & x_1 = 3 - 3x_2 + 2x_3 + 2x_5 \\
& x_4 = 2 - x_3 - x_5 \\
& x_6 = 1 - x_5 \\
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\end{aligned}
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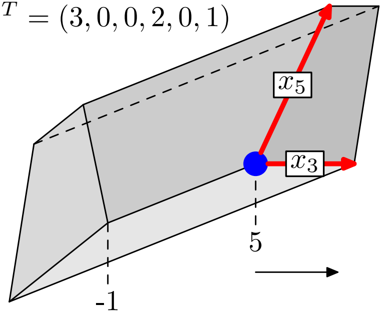

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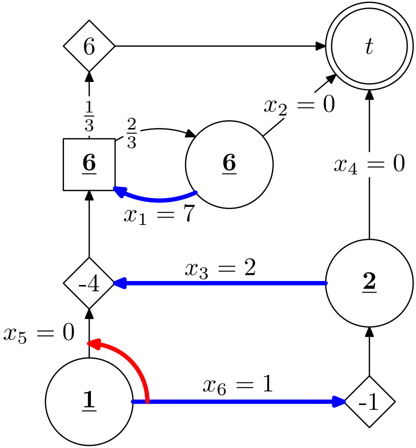

$$\begin{aligned} \max \quad & 9 - 6x_2 - 2x_4 + x_5 \\ \text{s.t.} \quad & x_1 = 7 - 3x_2 - 2x_4 \\ & x_3 = 2 - x_4 - x_5 \\ & x_6 = 1 - x_5 \\ & x_1, x_2, x_3, x_4, x_5, x_6 \geq 0 \end{aligned}$$

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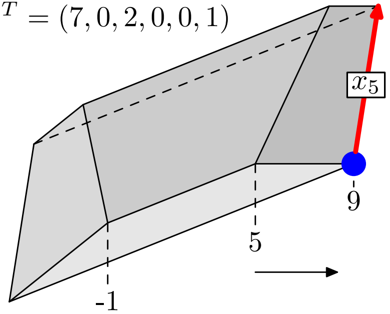

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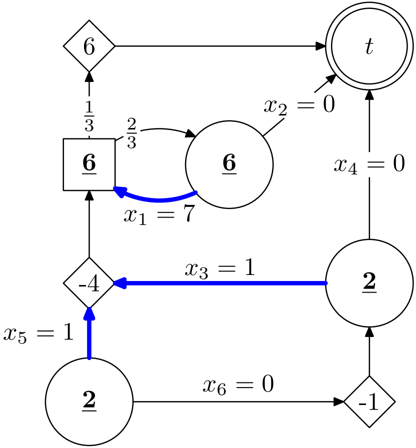

$$\begin{array}{llcl} \max & 10 - 6x_2 - 2x_4 - x_6 & & \\ \text{s.t.} & x_1 & = & 7 - 3x_2 - 2x_4 \\ & x_3 & = & 1 - x_4 + x_6 \\ & x_5 & = & 1 - x_6 \\ & x_1, x_2, x_3, x_4, x_5, x_6 \geq 0 & & \end{array}$$

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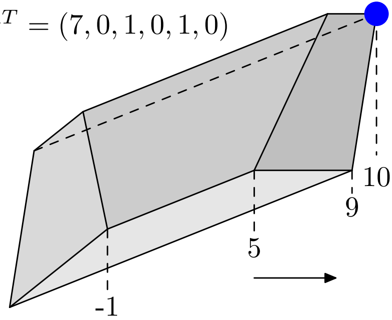

Lower bounds for pivoting rules utilizing MDPs

- To prove lower bounds for a pivoting rule for the simplex algorithm, we can prove lower bounds for the corresponding PolicyIteration algorithm for MDPs:
  - There is a one-to-one correspondence between policies and basic feasible solutions of the primal LP (P) for MDPs.
  - The simplex algorithm for the primal LP (P) is the special case of PolicyIteration, where only single improving switches are performed.

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- We next construct an exponential lower bound for Bland's rule as a warmup before sketching the $2^{\Omega(n^{1/4})}$ lower bound for RandomEdge by Friedmann, Hansen and Zwick (2011).

Bland's rule

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- When constructing a lower bound, we may pick a worst-case ordering of the indices.

Lower bound construction

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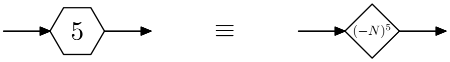
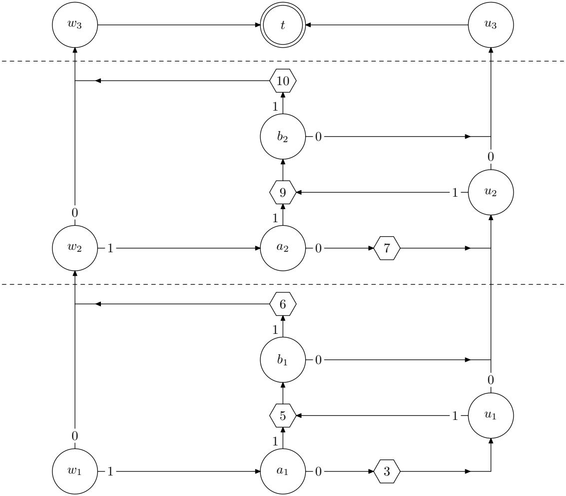
- We make use of exponentially growing rewards (and penalties): To get a higher reward the MDP is willing to sacrifice everything that has been built up so far.
- Notation: Integer priority p corresponds to reward $(-N)^p$, where $N = 3n + 1$.

$$\dots < 5 < 3 < 1 < 2 < 4 < 6 < \dots$$



Lower bound for Bland's rule



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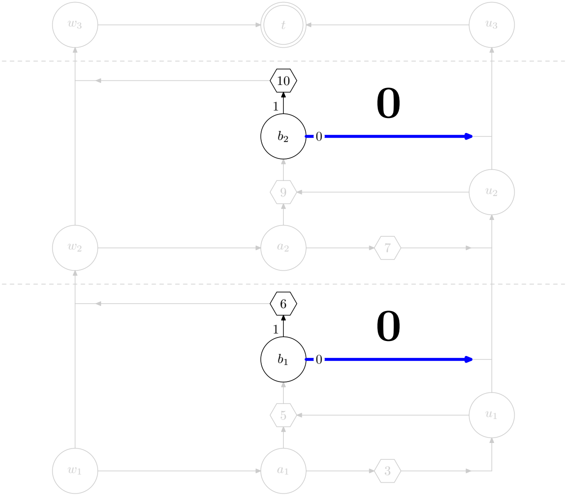

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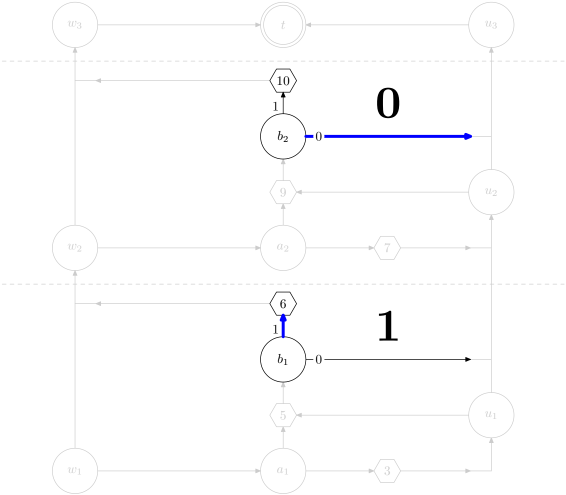

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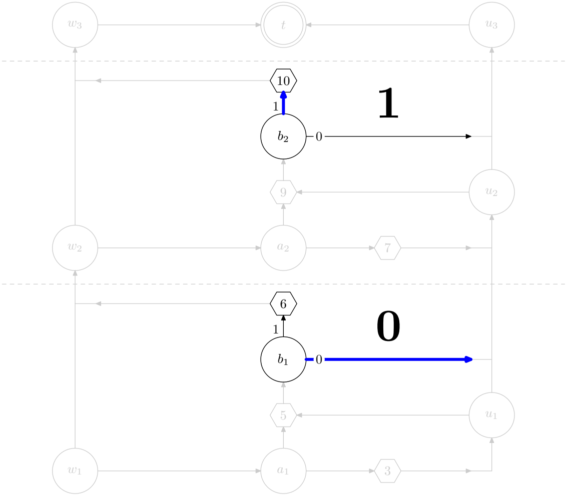

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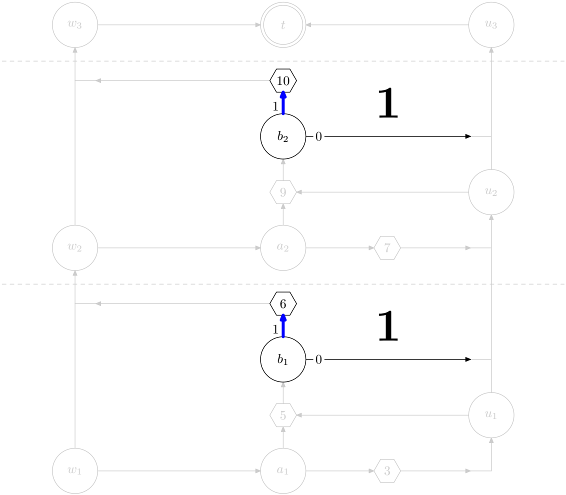

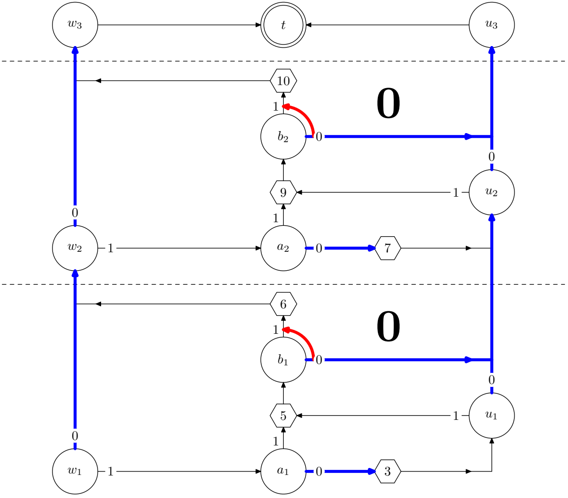

Order: $u_1^1, u_1^2, u_2^1, u_2^2, w_1^1, w_1^2, w_2^1, w_2^2, b_1^0, b_2^0, a_1^0, a_2^0, \underline{a_1^1, a_2^1, b_1^1, b_2^1}$

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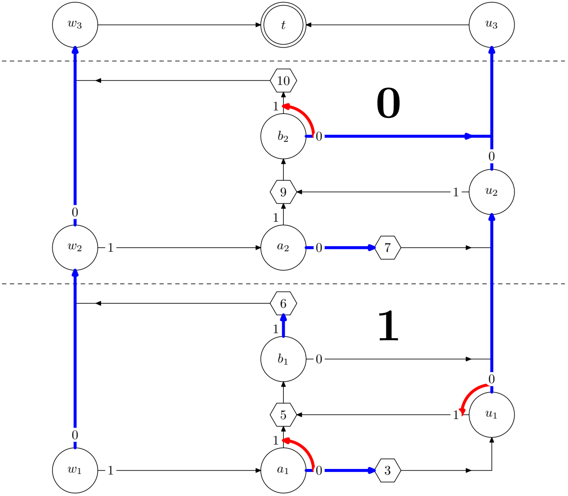

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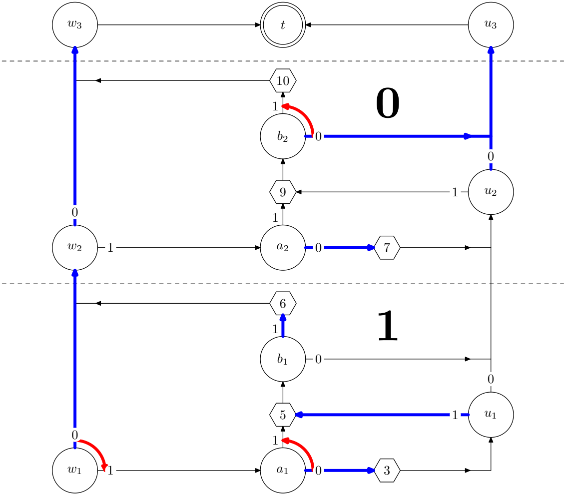
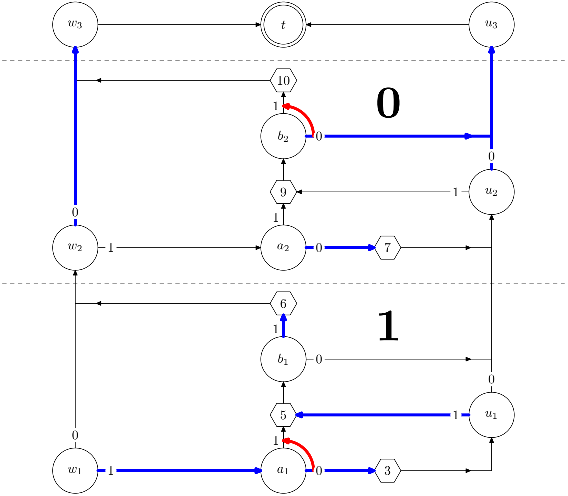
Lower bound for Bland's rule



Order: $u_1^1, u_1^2, u_2^1, u_2^2, w_1^1, w_1^2, w_2^1, w_2^2, b_1^0, b_2^0, a_1^0, a_2^0, \underline{a_1^1, a_2^1, b_1^1, b_2^1}$

Lower bound for BLAND'S RULE



Order: $u_1^1, u_1^2, u_2^1, u_2^2, w_1^1, w_1^2, w_2^1, w_2^2, b_1^0, b_2^0, a_1^0, a_2^0, \underline{a_1^1, a_2^1, b_1^1, b_2^1}$

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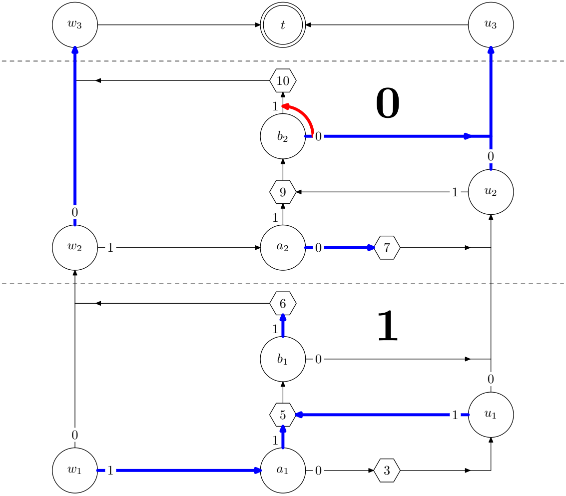
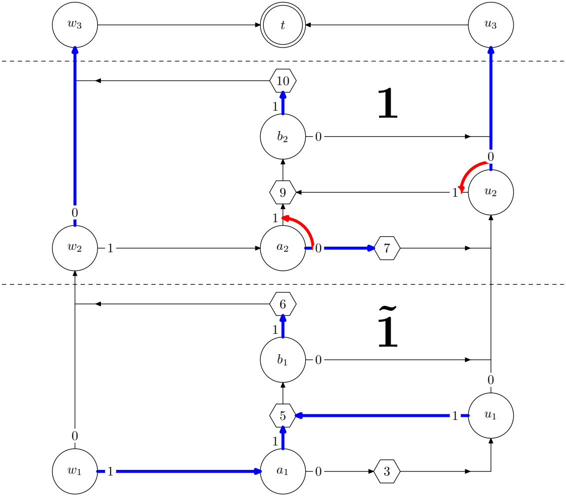


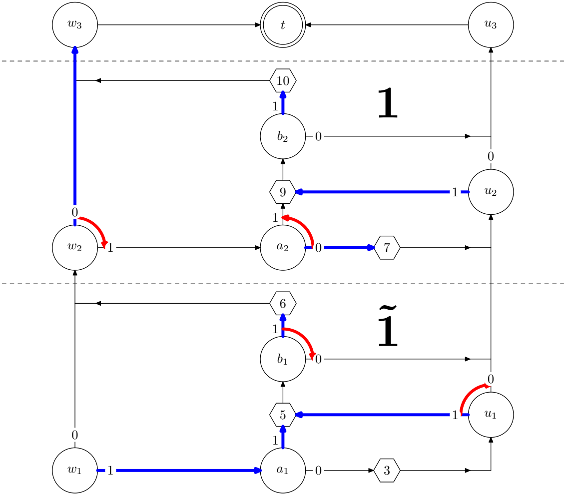
Order: $u_1^1, u_1^2, u_2^1, u_2^2, w_1^1, w_1^2, w_2^1, w_2^2, b_1^0, b_2^0, a_1^0, a_2^0, \underline{a_1^1, a_2^1, b_1^1, b_2^1}$

Lower bound for Bland's rule



Order: $u_1^1, u_1^2, u_2^1, u_2^2, w_1^1, w_1^2, w_2^1, w_2^2, b_1^0, b_2^0, a_1^0, a_2^0, \underline{a_1^1, a_2^1, b_1^1, b_2^1}$

Lower bound for BLAND'S RULE



Order: $u_1^1, u_1^2, u_2^1, u_2^2, w_1^1, w_1^2, w_2^1, w_2^2, b_1^0, b_2^0, a_1^0, a_2^0, \underline{a_1^1, a_2^1, b_1^1, b_2^1}$

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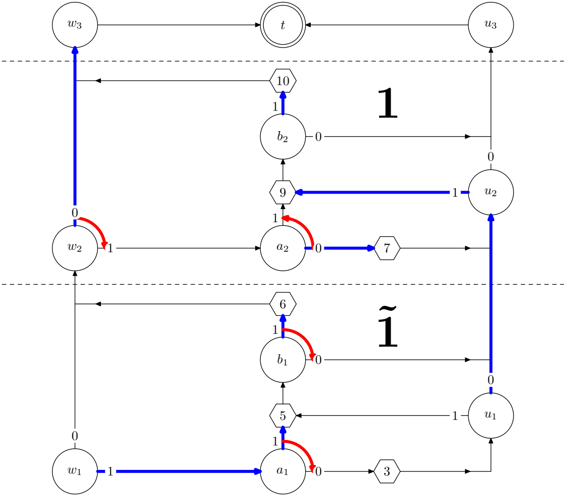

Order: $u_1^1, u_1^2, u_2^1, u_2^2, w_1^1, w_1^2, w_2^1, w_2^2, b_1^0, b_2^0, a_1^0, a_2^0, \underline{a_1^1, a_2^1, b_1^1, b_2^1}$

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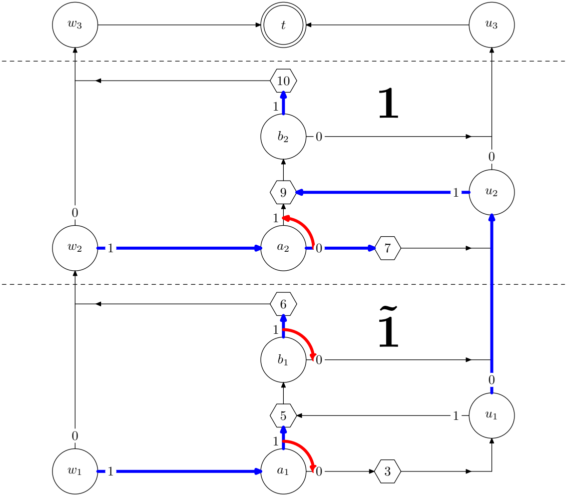

Order: $u_1^1, u_1^2, u_2^1, u_2^2, w_1^1, w_1^2, w_2^1, w_2^2, b_1^0, b_2^0, a_1^0, a_2^0, \underline{a_1^1, a_2^1, b_1^1, b_2^1}$

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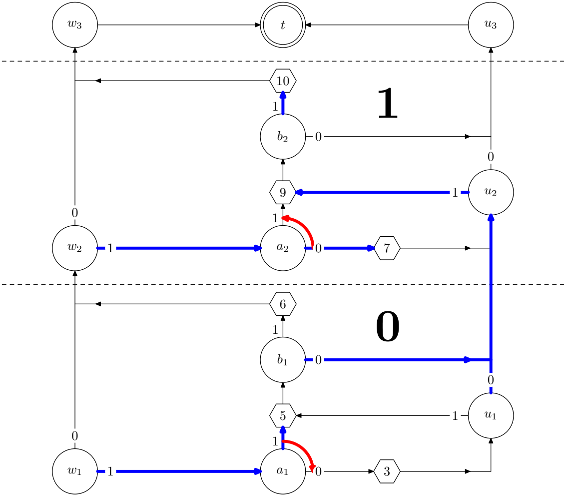

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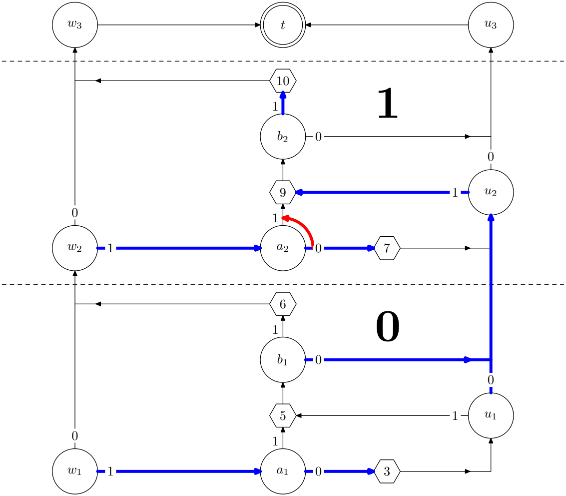

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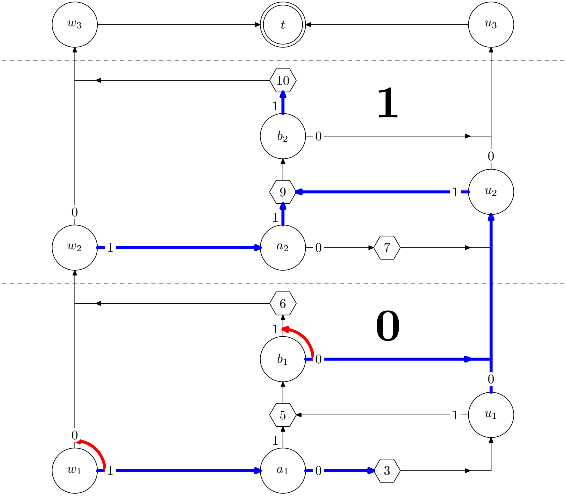
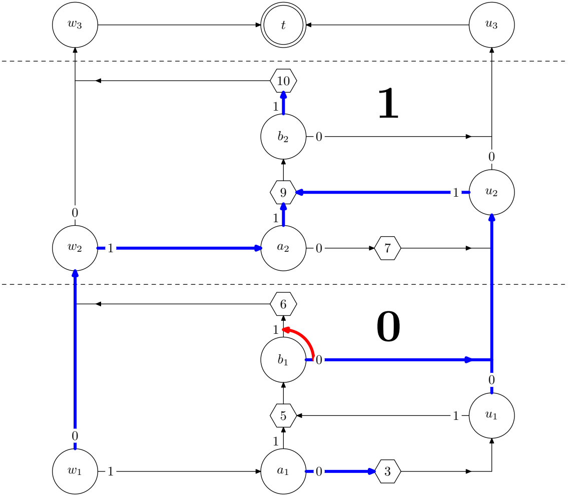


Order: $u_1^1, u_1^2, u_2^1, u_2^2, w_1^1, w_1^2, w_2^1, w_2^2, b_1^0, b_2^0, a_1^0, a_2^0, \underline{a_1^1, a_2^1, b_1^1, b_2^1}$

Lower bound for BLAND'S RULE

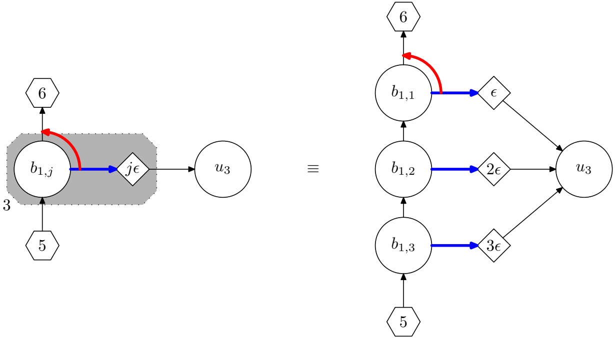


Order: $u_1^1, u_1^2, u_2^1, u_2^2, w_1^1, w_1^2, w_2^1, w_2^2, b_1^0, b_2^0, a_1^0, a_2^0, \underline{a_1^1, a_2^1, b_1^1, b_2^1}$

Lower bound for BLAND'S RULE

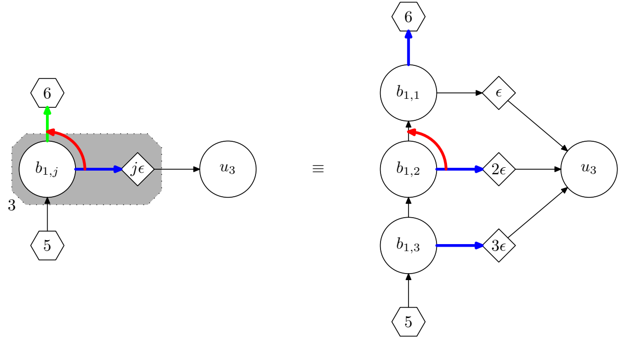
- Let k be the lowest unset bit. Incrementing the counter happened roughly through five phases:
  1. Make $b_k = 1$.
  2. Make $u_k = 1$ and $u_i = 0$, for $i < k$.
  3. Make $b_i = 0$ and $a_i = 0$, for $i < k$.
  4. Make $a_k = 1$.
  5. Make $w_k = 1$ and $w_i = 0$, for $i < k$.
- Only the last part of the ordering, involving a_i^1 and b_i^1 edges, was important.
- To implement a lower bound for RANDOMEDGE we start out with the same construction. We need a gadget to delay improving switches like a_i^1 and b_i^1, however.

Delaying events



- By replacing a vertex by a chain of vertices, a specific sequence of improving switches has to be performed to get the same effect as performing one improving switch originally.
- RANDOMEDGE performs uniformly random improving switches, and a longer sequence therefore gives a longer delay.

Delaying events



- By replacing a vertex by a chain of vertices, a specific sequence of improving switches has to be performed to get the same effect as performing one improving switch originally.
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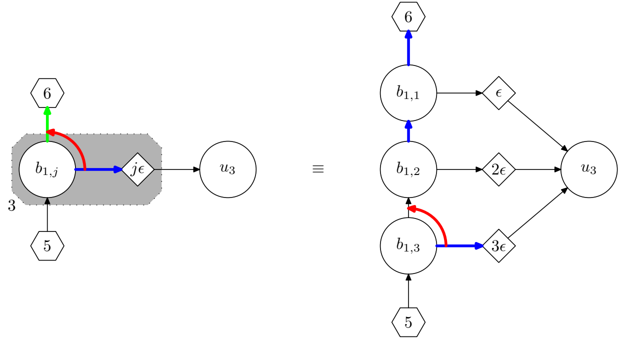
Delaying events



- By replacing a vertex by a chain of vertices, a specific sequence of improving switches has to be performed to get the same effect as performing one improving switch originally.
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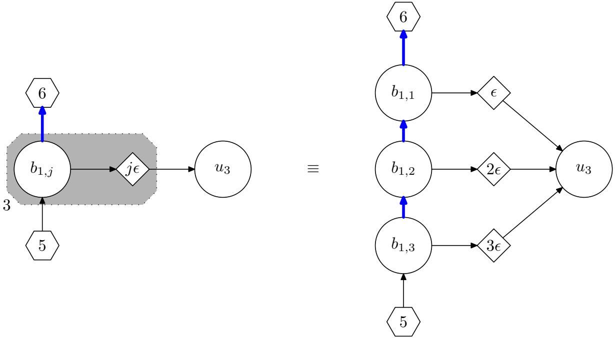
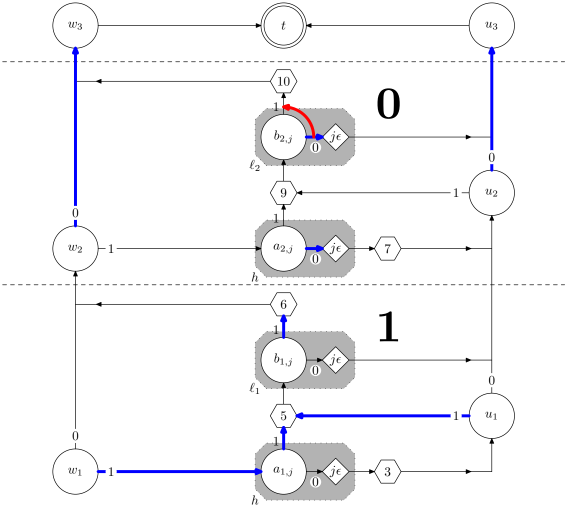
Delaying events



- By replacing a vertex by a chain of vertices, a specific sequence of improving switches has to be performed to get the same effect as performing one improving switch originally.
- RandomEdge performs uniformly random improving switches, and a longer sequence therefore gives a longer delay.

- Let k be the lowest unset bit. Incrementing the counter happens through five phases:

⇒ 1. Make $b_k = 1$.
2. Make $u_k = 1$ and $u_i = 0$, for $i < k$.
3. Make $b_i = 0$ and $a_i = 0$, for $i < k$.
4. Make $a_k = 1$.
5. Make $w_k = 1$ and $w_i = 0$, for $i < k$.

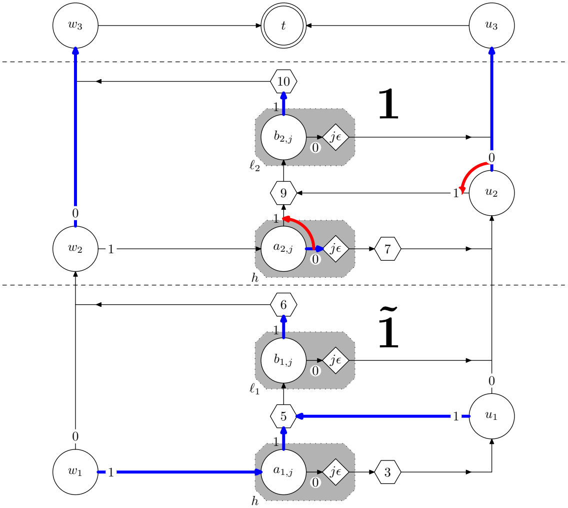


RandomEdge lower bound, first step

- Let k be the lowest unset bit. Incrementing the counter happens through five phases:

  1. Make $b_k = 1$.

  ⇒ 2. Make $u_k = 1$ and $u_i = 0$, for $i < k$.
  3. Make $b_i = 0$ and $a_i = 0$, for $i < k$.
  4. Make $a_k = 1$.
  5. Make $w_k = 1$ and $w_i = 0$, for $i < k$.



- Let k be the lowest unset bit. Incrementing the counter happens through five phases:

1. Make $b_k = 1$.
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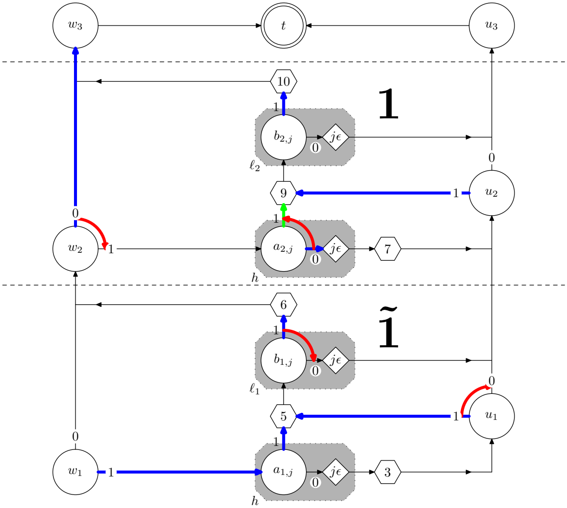

- Let k be the lowest unset bit. Incrementing the counter happens through five phases:

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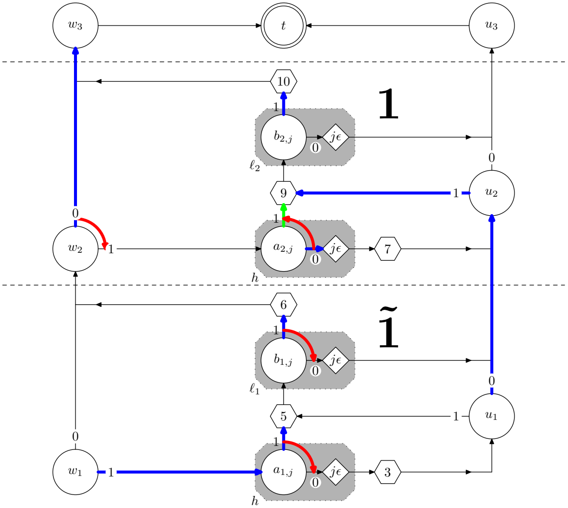
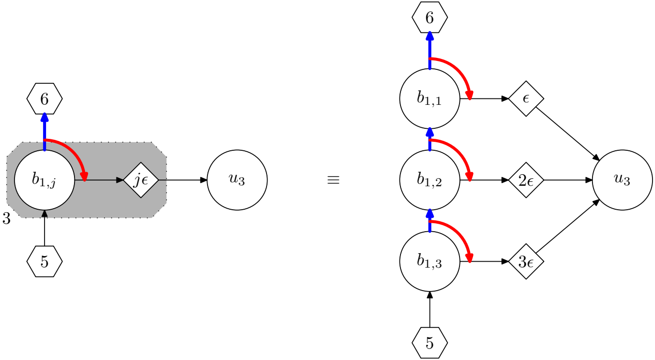
  5. Make $w_k = 1$ and $w_i = 0$, for $i < k$.



Fast resetting

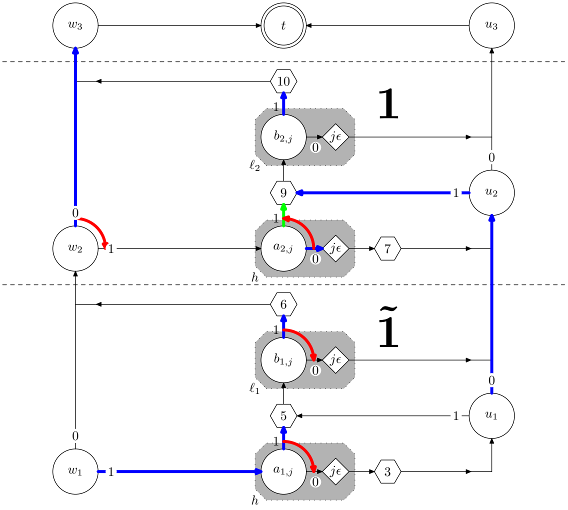


- Moving in the other directions happens much faster since all actions are improving switches simultaneously.

RandomEdge lower bound, first step

- Let k be the lowest unset bit. Incrementing the counter happens through five phases:

  1. Make $b_k = 1$.
  2. Make $u_k = 1$ and $u_i = 0$, for $i < k$.
  3. ⇒ Make $b_i = 0$ and $a_i = 0$, for $i < k$.
  4. Make $a_k = 1$.
  5. Make $w_k = 1$ and $w_i = 0$, for $i < k$.



RandomEdge lower bound, first step

- Let k be the lowest unset bit. Incrementing the counter happens through five phases:
  1. Make $b_k = 1$.
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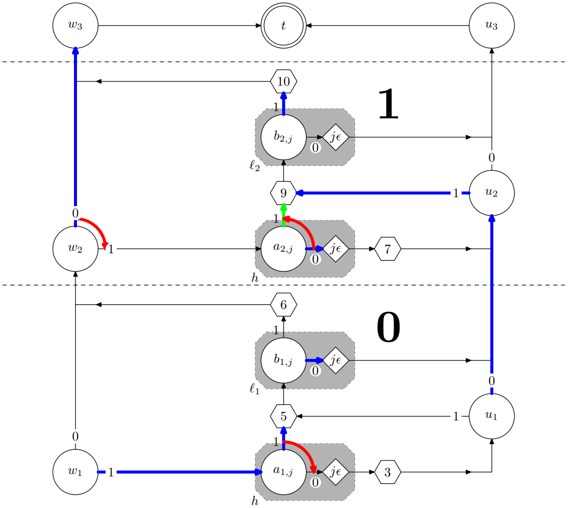
  4. Make $a_k = 1$.
  5. Make $w_k = 1$ and $w_i = 0$, for $i < k$.



RandomEdge lower bound, first step

- Let k be the lowest unset bit. Incrementing the counter happens through five phases:

  1. Make $b_k = 1$.
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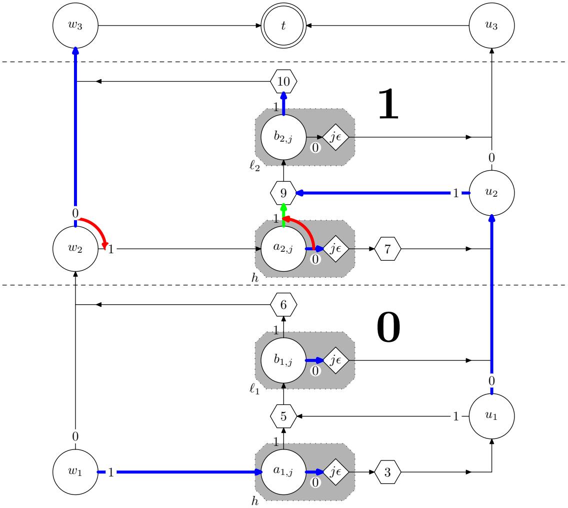
  3. Make $b_i = 0$ and $a_i = 0$, for $i < k$.
  4. ⇒ Make $a_k = 1$.
  5. Make $w_k = 1$ and $w_i = 0$, for $i < k$.



RandomEdge lower bound, first step

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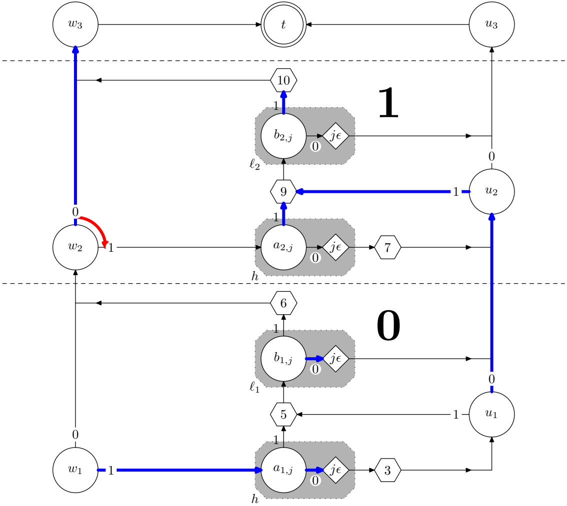

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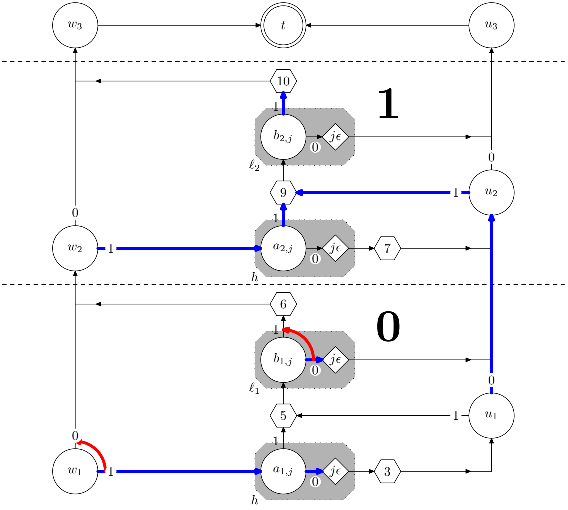

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$\Rightarrow$ 1. Make $b_k = 1$.
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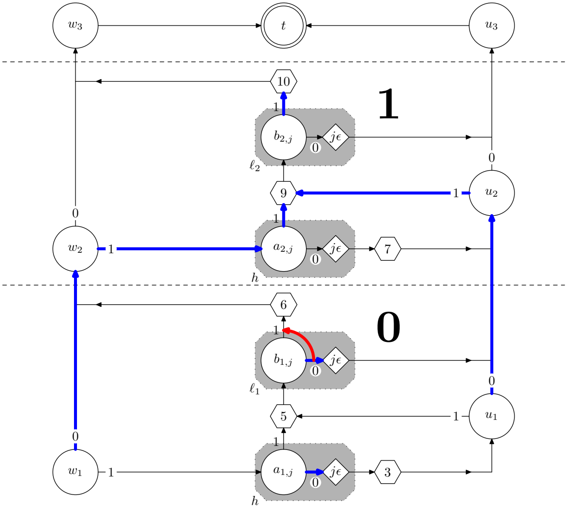
4. Make $a_k = 1$.
5. Make $w_k = 1$ and $w_i = 0$, for $i < k$.



Resetting partially set higher bits

- Problem: Higher bits get a head start in later counting steps. When resetting lower bits we must also reset higher bits that are partially set.

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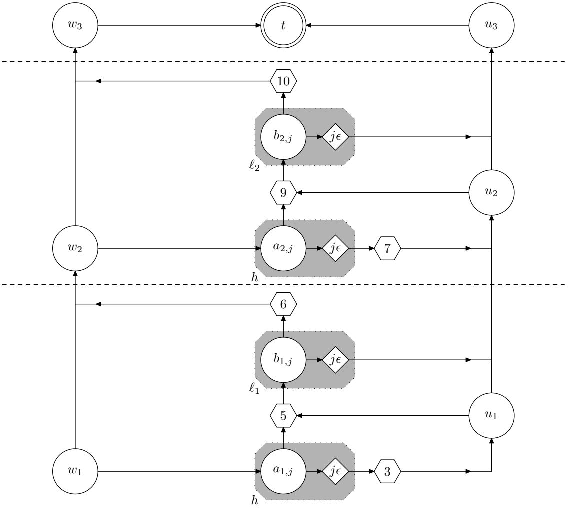
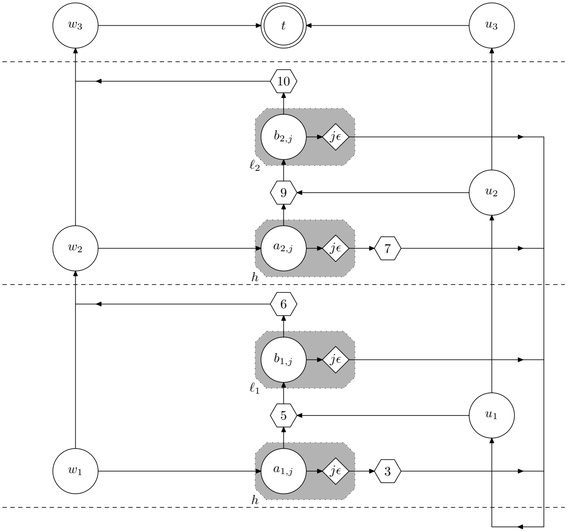
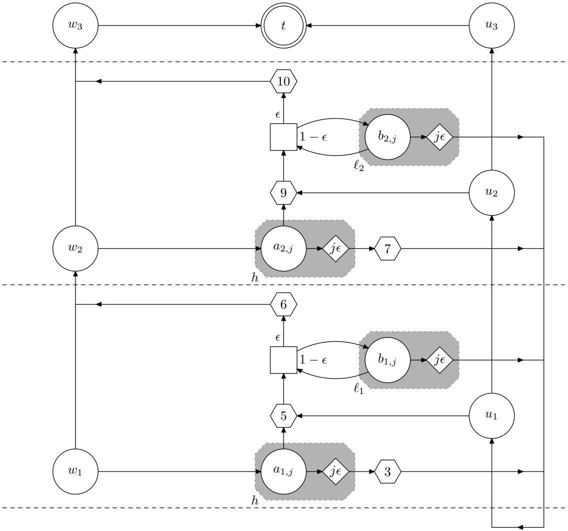
- No state can have direct access to a large reward: Introduce a stochastic action such that this happens only with an insignificant probability $\epsilon = N^{-(4n+8)}$.
- Resetting higher bits requires alternating behaviour: Introduce an additional chain of c_i vertices.

RandomEdge lower bound, full construction



RandomEdge lower bound, full construction

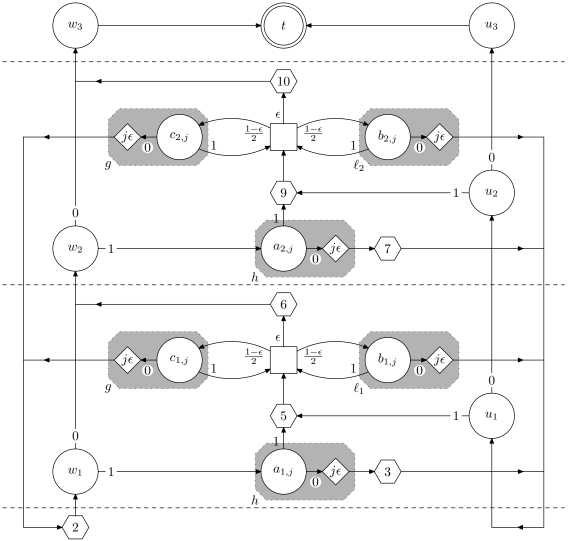



w_3
t
u_3
10
ϵ
$1-\epsilon$
$b_{2,j}$
$j\epsilon$
ℓ_2
9
u_2
w_2
$a_{2,j}$
$j\epsilon$
7
h
6
ϵ
$1-\epsilon$
$b_{1,j}$
$j\epsilon$
ℓ_1
5
u_1
w_1
$a_{1,j}$
$j\epsilon$
3
h

RandomEdge lower bound, full construction

- Incrementing the counter happens through seven phases:

  1. Make $b_k = 1$.
  2. Make $c_k = 1$.
  3. Make $u_k = 1$ and $u_i = 0$, for $i < k$.
  4. Make $b_i = 0$ and $a_i = 0$, for $i < k$. Reset b_i for unset bits $i > k$.
  5. Make $a_k = 1$.
  6. Make $w_k = 1$ and $w_i = 0$, for $i < k$.
  7. Reset c_i for all unset bits.



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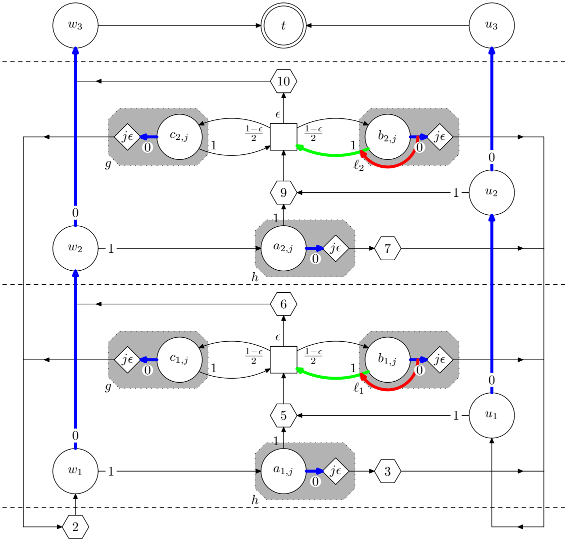

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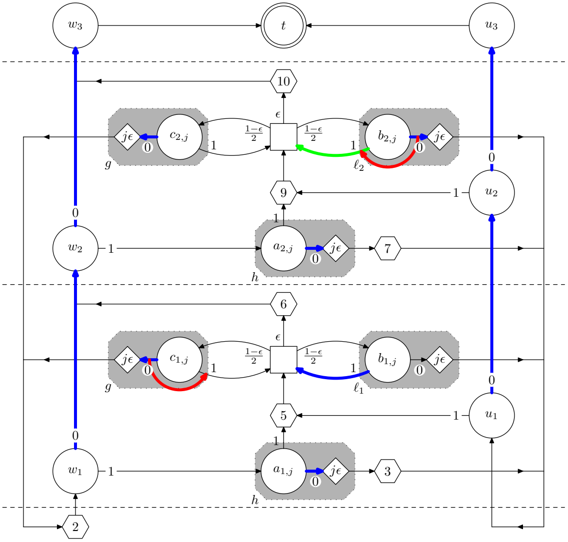

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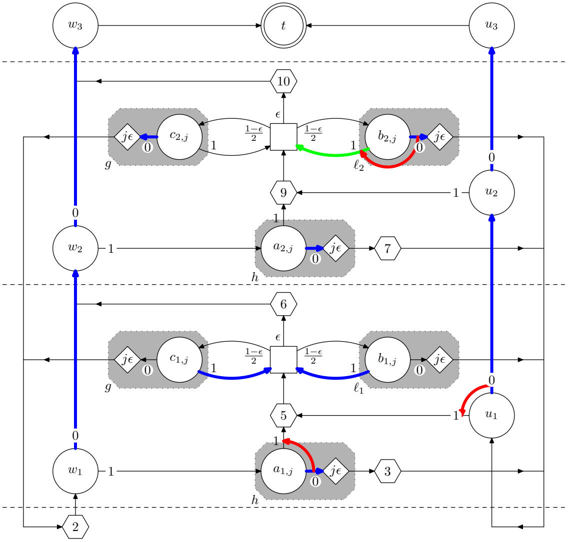

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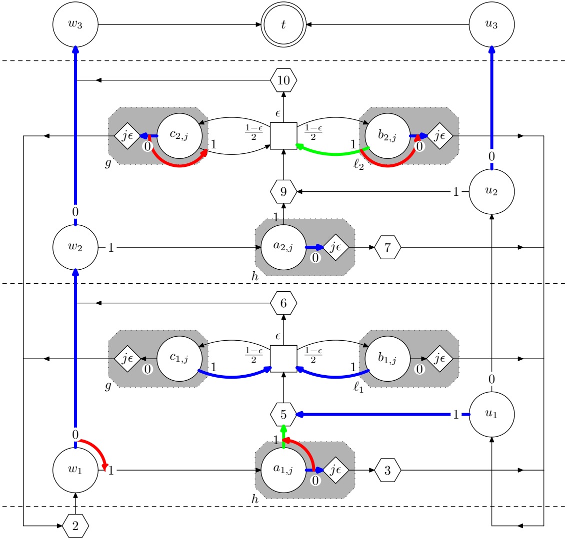

- Incrementing the counter happens through seven phases:

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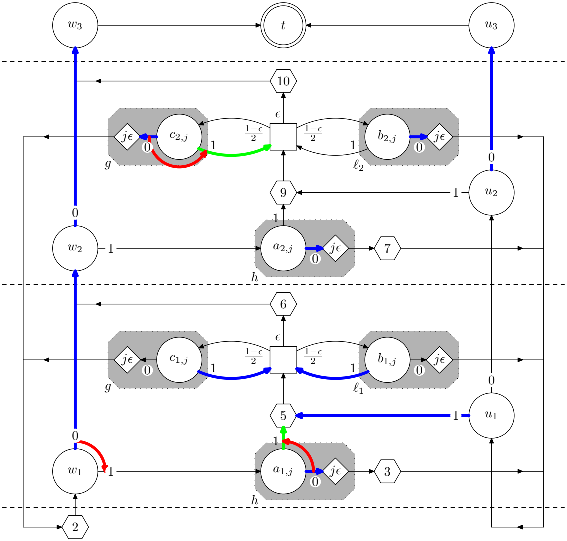

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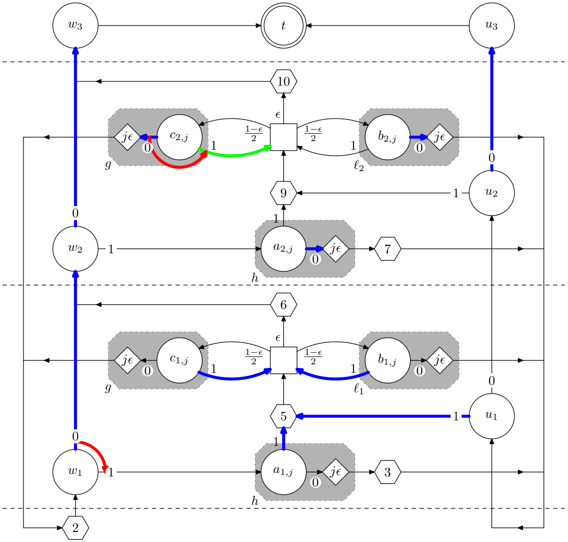

RandomEdge lower bound, full construction

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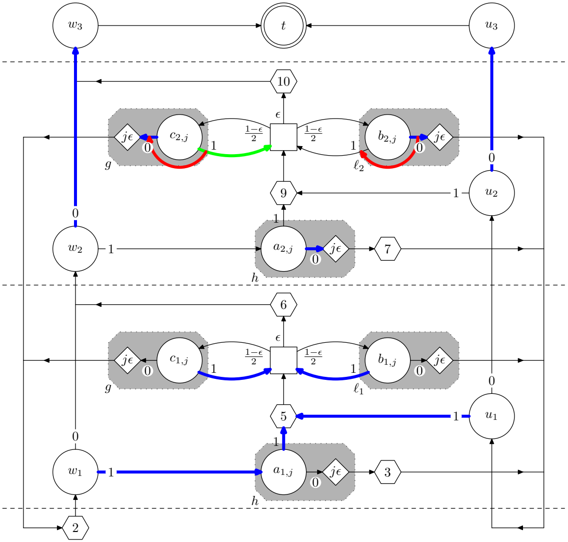

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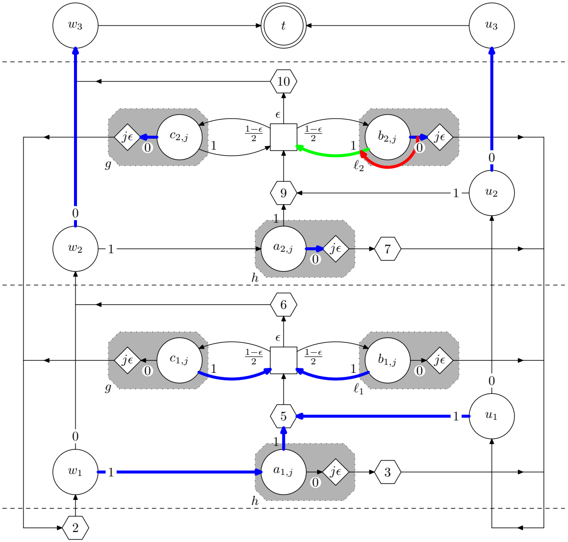

Competing chains

- The greatest challenge when setting the parameters is to make sure that the lowest unset bit is incremented next with high probability.
- The situation occurs when two chains b_i and b_{i+1} of lengths ℓ_i and ℓ_{i+1} are competing to change from 0 to 1.
- In both chains there is always exactly one improving switch, which means that the RandomEdge pivoting rule will pick either of them with equal probability.
- We bound the probability of failure with a Chernoff bound, and show that it suffices to set $\ell_i = \Theta(i^2 n)$.

RandomEdge lower bound

Theorem (Friedmann, Hansen and Zwick (2011))

The worst-case expected number of pivoting steps performed by RandomEdge *on linear programs with n equalities and $2n$ non-negative variables is* $2^{\Omega(n^{1/4})}$.

$$\begin{array}{ll} \text{maximize} & \sum_{i=1}^{n} \sum_{j=1}^{h} ((-N)^{4i-1} + j\epsilon) a^0_{i,j} + \\ & \sum_{i=1}^{n} ((-N)^{4i+1} + \epsilon(-N)^{4i+2})(a^1_{i,1} + u^1_i) + \\ & \sum_{i=1}^{n} (\epsilon(-N)^{4i+2})(b^1_{i,1} + c^1_{i,1}) + \\ & \sum_{i=1}^{n} \sum_{j=1}^{\ell_i} j\epsilon b^0_{i,j} + \sum_{i=1}^{n} \sum_{j=1}^{g} j\epsilon c^0_{i,j} \end{array}$$

subject to

$$\begin{array}{lrcl}
\forall 1 \le i \le n : & a^0_{i,h} + a^1_{i,h} & = & 1 + w^1_i \\
\forall 1 \le i \le n, \forall 1 \le j < h : & a^0_{i,j} + a^1_{i,j} & = & 1 + a^1_{i,j+1} \\
\forall 1 \le i \le n : & b^0_{i,\ell_i} + b^1_{i,\ell_i} & = & 1 + \frac{1-\epsilon}{2}\left(a^1_{i,1} + b^1_{i,1} + c^1_{i,1} + u^1_i\right) \\
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\forall 1 \le i \le n, \forall 1 \le j < g : & c^0_{i,j} + c^1_{i,j} & = & 1 + c^1_{i,j+1} \\
 & u^0_1 + u^1_1 & = & 1 + \sum_{i=1}^{n} \left(\sum_{j=1}^{h} a^0_{i,j} + \sum_{j=1}^{\ell_i} b^0_{i,j} \right) \\
\forall 2 \le i \le n : & u^0_i + u^1_i & = & 1 + u^0_{i-1} \\
 & w^0_1 + w^1_1 & = & 1 + \sum_{i=1}^{n} \sum_{j=1}^{g} c^0_{i,j} \\
\forall 2 \le i \le n : & w^0_i + w^1_i & = & 1 + w^0_{i-1} + \epsilon\left(a^1_{i,1} + b^1_{i,1} + c^1_{i,1} + u^1_i\right)
\end{array}$$

$$a^0_{i,j}, a^1_{i,j}, b^0_{i,j}, b^1_{i,j}, c^0_{i,j}, c^1_{i,j}, u^0_i, u^1_i, w^0_i, w^1_i \ge 0$$

Function $\textsc{RandomFacet}(G, \pi)$

```
if π contains all actions then
    return π
else
    Choose unused action a uniformly at random
    π' ← RandomFacet(G \ {a}, π)
    if a is improving switch w.r.t. π' then
        π'' ← π'[a]
        return RandomFacet(G, π'')
    else
        return π'
```

Function $\textsc{RandomFacet}(G, \pi, \varphi)$

```
if π contains all actions then
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    a ← argmin_{a ∈ A \ π} φ(a)
    π' ← RandomFacet(G \ {a}, π, φ)
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        π'' ← π'[a]
        return RandomFacet(G, π'', φ)
    else
        return π'
```

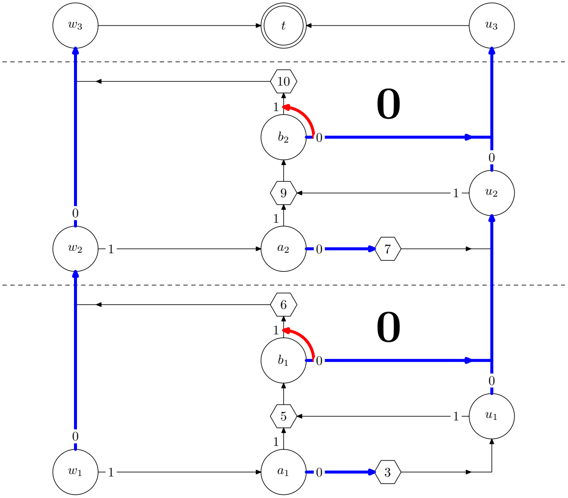
Lower bound for RandomFacet dropped to $2^{\tilde{\Omega}(n^{1/3})}$

- In Friedmann, Hansen and Zwick (SODA, 2011) we proved a $2^{\tilde{\Omega}(n^{1/2})}$ lower bound, for *parity games*, for the "modified RandomFacet algorithm" starting with a uniformly random permutation.
- We *incorrectly* believed, until less than three weeks ago, that by linearity of expectation the RandomFacet algorithm required the same expected number of steps. We now know that this is not true.
- Fortunately, using the same construction with different parameters we have been able to prove a $2^{\tilde{\Omega}(n^{1/3})}$ lower bound, which will be made public as soon as all the details have been written and verified.

Lower bound for RandomFacet dropped to $2^{\tilde{\Omega}(n^{1/3})}$

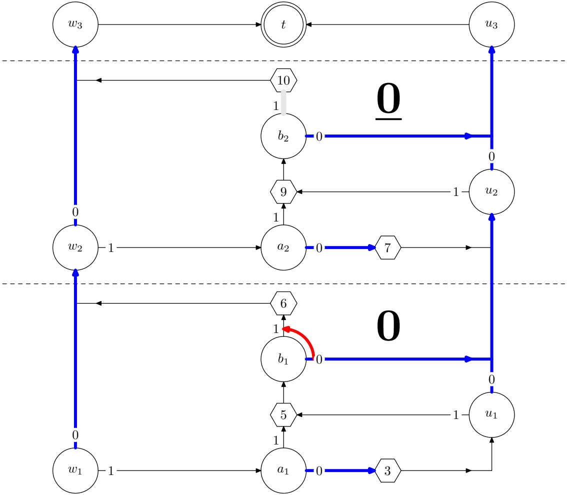
- Looking closer at the "modified RandomFacet algorithm", it turns out to be a dual, recursive variant of the Randomized Bland's rule.
- In Friedmann, Hansen and Zwick (STOC, 2011) we showed a simple transformation of our lower bound parity games to MDPs, thereby getting lower bounds for the simplex algorithm. This transformation remains the same.
  - The main result of this paper was the $2^{\Omega(n^{1/4})}$ lower bound for RandomEdge which is unaffected.

Simplified RandomFacet lower bound



Simplified RandomFacet lower bound

- The RandomFacet algorithm picks a random edge, here b_2^1, and removes it, thereby disabling the bit.



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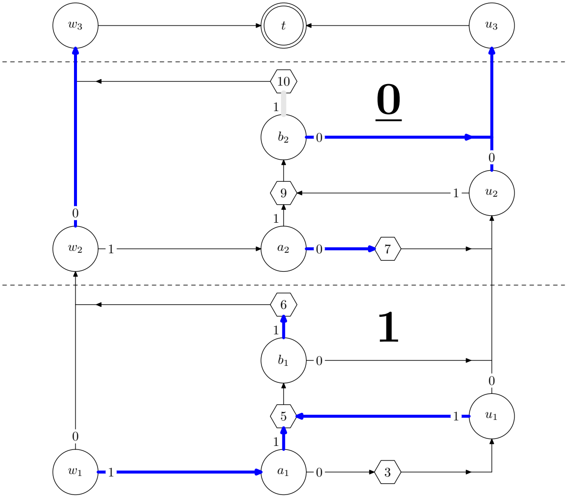

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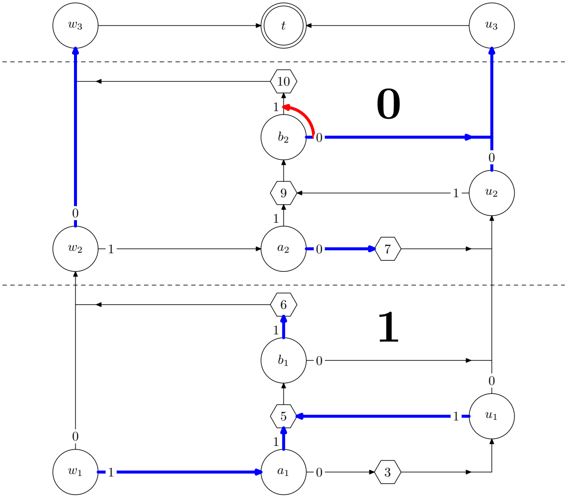

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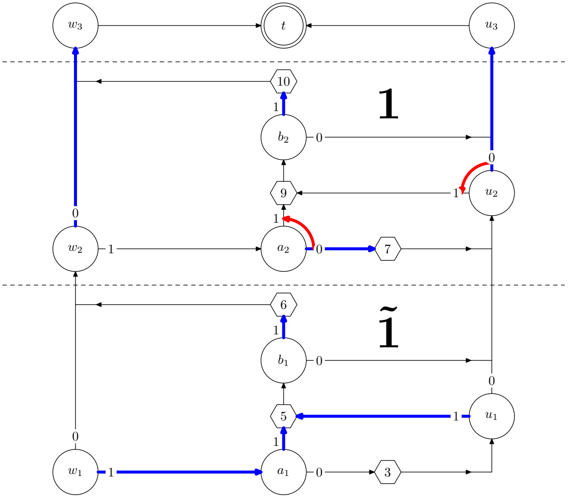

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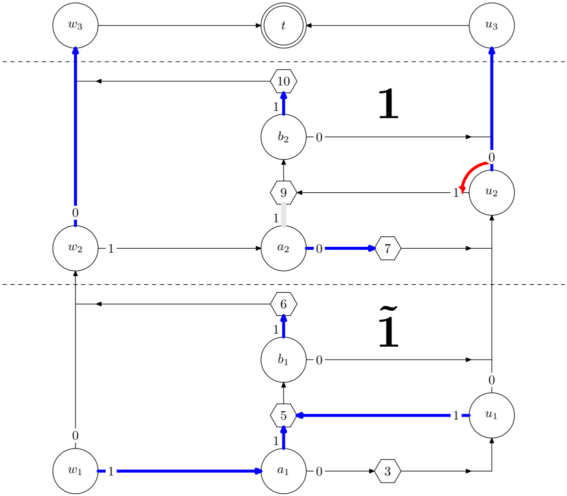

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- A new random edge, a_2^1, is removed.
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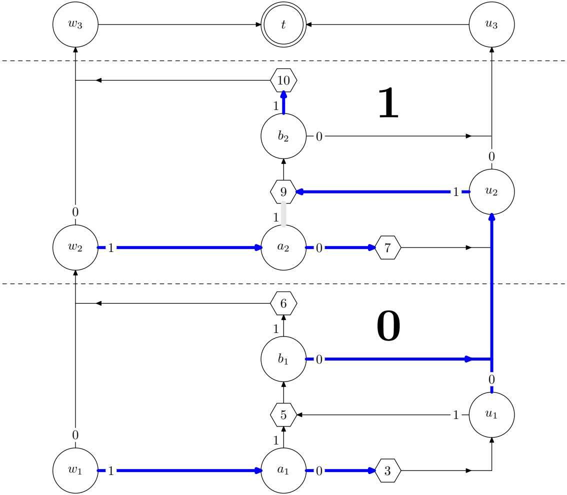

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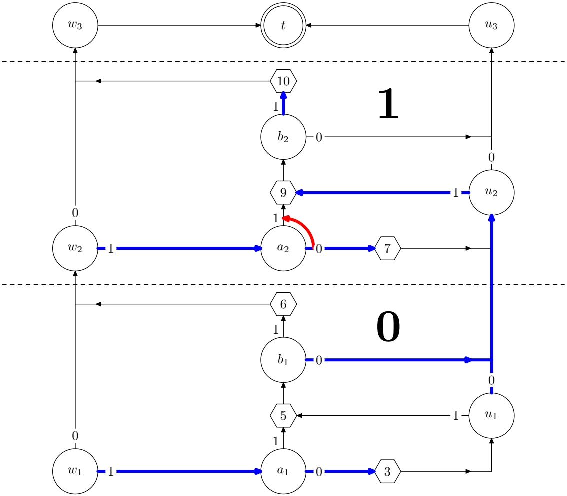

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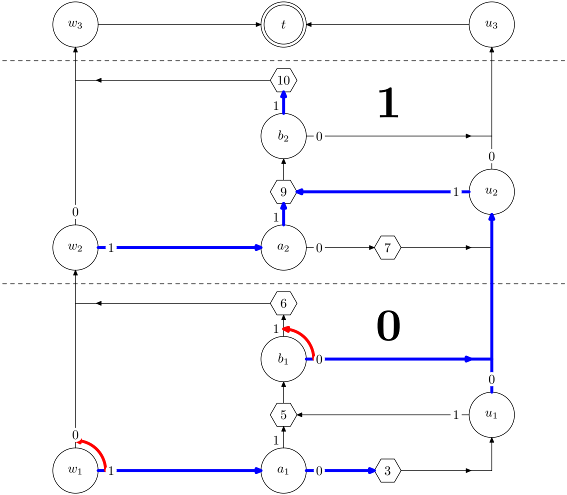

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- Expected number of increments:

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- Solving the recurrence gives: $f(n) = 2^{\Theta(\sqrt{n})}$

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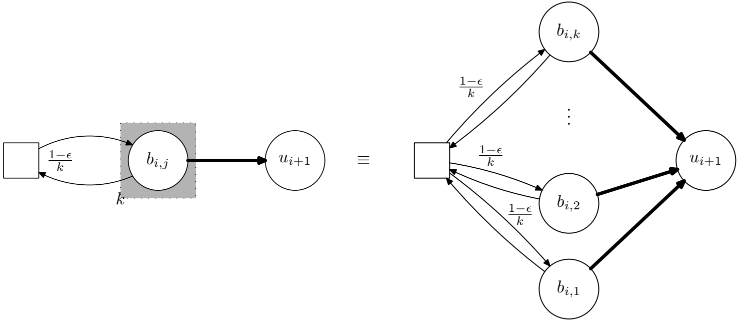
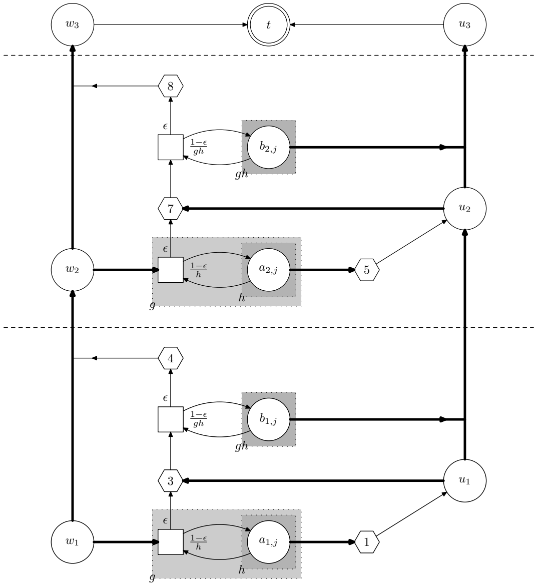
- The only *bad* permutation for the "modified RandomFacet pivoting rule" is then: $aa \ldots abb \ldots b$
- The probability of choosing a bad permutation is $\frac{(k!)^2}{(2k)!} \leq \frac{1}{2^k}$.

Different gadgets



- We use different gadgets to ensure that we get the correct behaviour with high probability.
- For the "modified RandomFacet pivoting rule" this only increases the number of states and actions by a polylogarithmic factor.
- For the real RandomFacet pivoting rule the increase needs to be a factor $\tilde{O}(\sqrt{n})$.

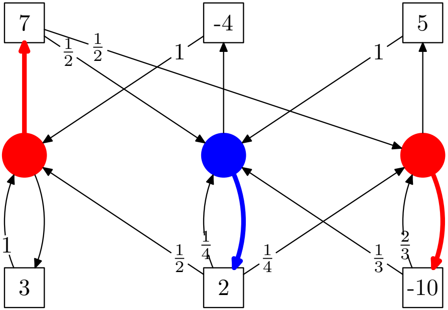
RandomFacet lower bound construction



Overview

- **Lecture 1:**
  - Introduction to linear programming and the simplex algorithm.
  - Pivoting rules.
  - The RandomFacet pivoting rule.
- **Lecture 2:**
  - The Hirsch conjecture.
  - Introduction to Markov decision processes (MDPs).
  - Upper bound for the LargestCoefficient pivoting rule for MDPs.
- **Lecture 3:**
  - Lower bounds for pivoting rules utilizing MDPs. Example: Bland's rule.
  - Lower bound for the RandomEdge pivoting rule.
  - Abstractions and related problems.

2-player turn-based stochastic games (2TBSGs)



- A 2TBSG is an MDP where the set of states is partitioned into two sets: $S_1 \cup S_2 = S$.
  - S_1 is controlled by player 1, the maximizer.
  - S_2 is controlled by player 2, the minimizer.
- A strategy π_1 (or π_2) for player 1 (or player 2) is a choice of an action for each state $i \in S_1$ (or $i \in S_2$).

2-player turn-based stochastic games (2TBSGs)

- A strategy profile $\pi = (\pi_1, \pi_2)$ is a pair of strategies, defining a Markov chain with rewards.
- The value vector for discounted 2TBSGs is again defined as:

$$v_\pi = (I - \gamma P_\pi)^{-1} c_\pi$$

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- A best response from player 1, $\pi_1(\pi_2)$, is defined analogously.

2-player turn-based stochastic games (2TBSGs)

- π_1^* and π_2^* are optimal if:

$$\forall \pi_1 : \quad v_{\pi_1^*, \pi_2(\pi_1^*)} \geq v_{\pi_1, \pi_2(\pi_1)}$$
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- Note that the decision problem corresponding to solving 2TBSGs is in **NP** $\cap$ *co***NP**, since an optimal strategy profile is a witness for both *yes* and *no* answers. The problem is not known to be in **P**.

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- We again say that $a \in A_i$, for $i \in S_1$, is an **improving switch** for player 1 w.r.t. π iff:

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- Similarly, $a \in A_i$, for $i \in S_2$, is an improving switch for player 2 w.r.t. π iff:

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- Similarly, $a \in A_i$, for $i \in S_2$, is an improving switch for player 2 w.r.t. π iff:

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- The vector of reduced costs for a strategy profile π is again defined as:

$$\bar{c}^\pi = c - (J - \gamma P) v_\pi$$

2-player turn-based stochastic games (2TBSGs)

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- Note that (π_1^*, π_2^*) is a Nash equilibrium iff there are no improving switches.

Function StrategyIteration(π_1)

while $\exists$ *improving switch w.r.t.* $(\pi_1, \pi_2(\pi_1))$ **do**

  Update π_1 by performing improving switches

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- Howard's algorithm can be naturally extended to 2TBSGs by choosing:

$$\forall i \in S_1 : \quad \pi_1(i) \leftarrow \operatorname*{argmax}_{a \in A_i} \bar{c}_a^{\pi_1, \pi_2(\pi_1)}$$

Non-discounted MDPs and 2TBSGs

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- MDPs (and 2TBSGs) satisfying the stopping condition are a special case of non-discounted MDPs (and 2TBSGs). See, e.g., Puterman (1994).

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  - Note that the mean of the rewards of a cycle is positive iff the parity of the largest priority is even.
- There is no known polynomial time algorithm for solving parity games.

A few results about StrategyIteration

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- Hansen, Miltersen and Zwick (2011): $O(\frac{m}{1-\gamma} \log \frac{n}{1-\gamma})$ iterations for discounted 2TBSGs with n states and m actions.

2TBSGs are LP-type problems

- Ludwig (1995), Halman (2007): 2TBSGs are LP-type problems.
  - Let H be the set of actions for player 1, and let ω map a subgame to the sum of its optimal values. Bases are strategies for player 1.
  - *Monotonicity*: More available actions only increases the value.
  - *Locality*: If $F \subseteq G \subseteq H$ and $-\infty < \omega(F) = \omega(G)$, then F and G share optimal strategies, and if an added action $h \in H$ is an improving switch for one then it also is for the other.

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- Hence, the dual RANDOMFACET algorithm can be used to solve 2TBSGs, and, in fact, it gives the best known bound for solving the non-discounted problem.

Unique sink orientations of cubes

- MDPs and 2TBSGs with two actions per state can be described abstractly by acyclic unique sink orientations (AUSOs) of hypercubes:
  - Strategies for player 1 map to vertices of the cube, and improving switches define an orientation of the edges such that in each face there is a unique sink.
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    - An algorithm can evaluate vertices of the cube to learn the orientation of the adjacent edges, and the goal is to find the unique sink of the entire cube.
- Szabó and Welzl (2001) introduced the FibonacciSeesaw algorithm for solving n dimensional unique sink orientations with F_{n+2} vertex evaluations.

More extensions of 2TBSGs

- There is no known way to formulate 2TBSGs as linear programs.
- Gärtner and Rüst (2005), and Jurdziński and Ravani (2008) showed how to formulate 2TBSGs with two actions per state as P-matrix linear complementarity problems.

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- P-matrix linear complementarity problems are also generalized by USOs, but not by AUSOs.
- Solving P-matrix linear complementarity problems, as well as 2TBSGs, is known to be in $\mathbf{PPAD} \cap \mathbf{PLS}$. Daskalakis and Papadimitriou (2011) suggested a new complexity class **CLS** (continuous local search) for capturing these and other problems.